Spectral Sequences

Cloudifold

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0 Towards to Constructions

0.1 Homology of length-2-filtration of chain complexes

Suppose we have a chain complex of **finite dimensional vector spaces**

$$C_* = \{\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots\}_{n\in\mathbb{Z}}$$

If it is **graded**

$$C_* \cong \bigoplus_{k=1}^{m} C_{*,k}$$

we can split differential $d_n \cong \bigoplus_{k=1}^{m} d_{n,k}$ then we can break up homology of C_* into smaller pieces: $H_n(C_*) \cong \bigoplus_{k=1}^{m} H_n(C_{*,k})$.

Now suppose we have a **filtration** of C_* **as complexes**: $0 = F_0C_* \hookrightarrow F_1C_* \hookrightarrow F_2C_* = C_*$ which means $\forall n \in \mathbb{Z},\ d_n(F_qC_n) \subseteq F_qC_{n-1}$.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
0 = F_0C_{n+1} & \hookrightarrow & F_1C_{n+1} & \hookrightarrow & F_2C_{n+1} = C_{n+1} \\
\downarrow & & \downarrow{\scriptstyle d_{n+1}|_{F_1C_{n+1}}} & & \downarrow{\scriptstyle d_{n+1}} \\
0 = F_0C_n & \hookrightarrow & F_1C_n & \hookrightarrow & F_2C_n = C_n \\
\downarrow & & \downarrow{\scriptstyle d_n|_{F_1C_n}} & & \downarrow{\scriptstyle d_n} \\
0 = F_0C_{n-1} & \hookrightarrow & F_1C_{n-1} & \hookrightarrow & F_2C_{n-1} = C_{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

For each n we can decomposite C_n into direct sum

$$C_n \cong \frac{C_n}{F_1C_n} \oplus F_1C_n = G_2C_n \oplus G_1C_n$$

where G_pC_n is the **associated graded object** $G_pC_* := \dfrac{F_pC_*}{F_{p-1}C_*}$.

Since $d_n(F_1C_n) \subseteq F_1C_{n-1}$, d_n on the component $\dfrac{C_n}{F_1C_n}$ is well-defined. Furthermore, d_n induces a map $d_n^0 : G_2C_n \oplus G_1C_n \to G_2C_{n-1} \oplus G_1C_{n-1}$

Real difficulty of decompositing **chain complex** C_* into chain complexes $G_2C_* \oplus G_1C_*$ is that for $c \in d_n^{-1}(F_1C_{n-1}) \subseteq C_n$, $d_n(c) \neq d_{n,2}^0 \oplus d_{n,1}^0(c)$:

$$C_n = \frac{C_n}{F_1C_n} \oplus F_1C_n \xrightarrow{d_n} \frac{C_{n-1}}{F_1C_{n-1}} \oplus F_1C_{n-1}$$
$$c = (a,b) \longmapsto (0, d_n(a,b)) \neq (0, d_{n,1}^0(b))$$

This difficulty would make $\operatorname{Im} d_n$ larger than $\operatorname{Im}(d_{n,2}^0 \oplus d_{n,1}^0)$ and $\ker d_n$ smaller than $\ker(d_{n,2}^0 \oplus d_{n,1}^0)$. Intuitively, we plot d as:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \dfrac{C_{n+1}}{F_1C_{n+1}} & \xrightarrow{d_{n+1,2}^0} & \dfrac{C_n}{F_1C_n} & \xrightarrow{d_{n,2}^0} & \dfrac{C_{n-1}}{F_1C_{n-1}} & \longrightarrow & \cdots \\
& & \oplus & \searrow{\scriptstyle |_{d_{n+1}^{-1}(F_1C_n)}} & \oplus & \searrow{\scriptstyle |_{d_n^{-1}(F_1C_{n-1})}} & \oplus & & \\
\cdots & \longrightarrow & F_1C_{n+1} & \xrightarrow[d_{n+1,1}^0]{} & F_1C_n & \xrightarrow[d_{n,1}^0]{} & F_1C_{n-1} & \longrightarrow & \cdots
\end{array}$$

Where the diagonal "maps" (call them D^1_n) are defined only on $\ker d^0_{*,2} = \dfrac{d_*^{-1}(F_1 C_{*-1})}{F_1 C_*}$.

The real homology of $C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$ is

$$\frac{\ker(d^0_{n,2} \oplus d^0_{n,1}) \cap \ker(D^1_n)}{\operatorname{Im}(d^0_{n+1,2} \oplus d^0_{n+1,1}) + \operatorname{Im}(D^1_{n+1})} = \frac{\ker d^0_{n,2} \cap \ker D^1_n}{\operatorname{Im} d^0_{n+1,2}} \oplus \frac{\ker d^0_{n,1}}{\operatorname{Im} d^0_{n+1,1} + \operatorname{Im} D^1_{n+1}}$$

Naturally, we want it be some homology of

$$\frac{\ker(d^0_{n,2} \oplus d^0_{n,1})}{\operatorname{Im}(d^0_{n+1,2} \oplus d^0_{n+1,1})} = \frac{\ker d^0_{n,2}}{\operatorname{Im} d^0_{n+1,2}} \oplus \frac{\ker d^0_{n,1}}{\operatorname{Im} d^0_{n+1,1}}$$

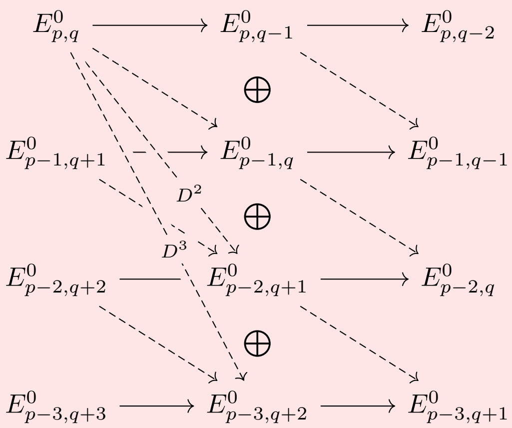
Since $D^1_n \circ d^0_{n+1,2} = 0$ and $d^0_{n,1} \circ D^1_n$, the diagonal arrows D^1_n define **real** maps $d^1_n : \dfrac{\ker d^0_{n,2}}{\operatorname{Im} d^0_{n+1,2}} \to \dfrac{\ker d^0_{n+1,1}}{\operatorname{Im} d^0_{n,1}}$. Now we get $H_n(C_*) = \ker d^1_n \oplus \operatorname{coker} d^1_{n+1}$.

0.2 Homology of length-n-filtration of chain complexes

Suppose we have a **filtration** of length n of C_*:

$$0 = F_0 C_* \hookrightarrow F_1 C_* \hookrightarrow \cdots \hookrightarrow F_{n-1} C_* \hookrightarrow F_n C_* = C_*$$

We define $E^0_{p,q} := G_p C_{p+q} = \dfrac{F_p C_{p+q}}{F_{p-1} C_{p+q}}$ and $d^0_{p,q} : E^0_{p,q} \to E^0_{p,q-1}$ is the map induced by $d_{p+q} : C_{p+q} \to C_{p+q-1}$. In this case, d is more complicated:



The "map" $D^2 : \dfrac{F_p C_{p+q}}{F_{p-1} C_{p+q}} \to \dfrac{F_{p-2} C_{p-1+q}}{F_{p-3} C_{p-1+q}}$ comes from restricting $d|_{F_p C_{p+q}}$ on $\ker D^1 = d^{-1}(F_{p-2} C_{p-1+q}) \cap F_p C_{p+q}$.

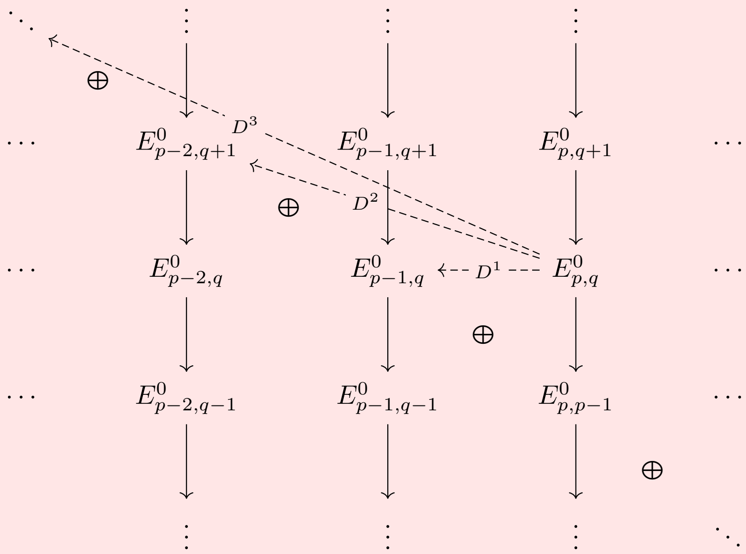
Inductively, we obtain "map" $D^r : \dfrac{F_p C_{p+q}}{F_{p-1} C_{p+q}} \to \dfrac{F_{p-r} C_{p-1+q}}{F_{p-r-1} C_{p-1+q}}$ defined by restricting d on $\ker D^{r-1} = d^{-1}(F_{p-r} C_{p-1+q}) \cap F_p C_{p+q}$.

Intuitively, we want:

- $\ker d \cap E^0_{p,q} = \ker d^0 \cap (\bigcap_r \ker D^r)$
- $\operatorname{Im} d \cap E^0_{p,q} = \operatorname{Im} d^0 + (\sum_r \operatorname{Im} D^r)$

If we have identities above, we could compute $H_n(C) = \dfrac{\ker d \cap \bigoplus_{p+q=n} E^0_{p,q}}{\operatorname{Im} d \cap \bigoplus_{p+q=n} E^0_{p,q}} = \bigoplus_{p+q=n} \dfrac{\ker d^0 \cap (\bigcap_r \ker D^r)}{\operatorname{Im} d^0 + (\sum_r \operatorname{Im} D^r)}$.

We plot these "maps" under our indexing convention:



Inspired by the length 2 case, we take iterated homology to to convert D^r into **well-defined** maps. Define r**-almost cycles** $Z^r_{p,q}$ and r**-almost boundaries** $B^r_{p,q}$ by:

$$\begin{aligned}
Z^r_{p,q} :=& \frac{\{c \in F_p C_{p+q} \mid d(c) \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q}} \\
=& \frac{d^{-1}(F_{p-r} C_{p+q-1}) \cap F_p C_{p+q}}{F_{p-1} C_{p+q}} \\
=& \text{intuitively } \ker D^{r-1} \\
B^r_{p,q} :=& \frac{d(F_{p+r-1} C_{p+q+1}) \cap F_p C_{p+q}}{F_{p-1} C_{p+q}} \\
=& \text{intuitively } \operatorname{Im} D^{r-1}
\end{aligned}$$

On $E^r_{p,q} := \dfrac{Z^r_{p,q}}{B^r_{p,q}}$, D^r becomes a well-defined map

$$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$
$$\frac{d^{-1}(F_{p-r} C_{p+q-1}) \cap F_p C_{p+q}}{d(F_{p+r-1} C_{p+q+1}) \cap F_p C_{p+q} + F_{p-1} C_{p+q}} \xrightarrow{d|} \frac{d^{-1}(F_{p-2r} C_{p+q-2}) \cap F_{p-r} C_{p+q-1}}{d(F_{p-1} C_{p+q}) \cap F_{p-r} C_{p+q-1} + F_{p-r-1} C_{p+q-1}}$$

And is still differential.
Our intuition $Z^r = \ker D^{r-1}$ and $B^r = \operatorname{Im} D^{r-1}$ are now strict identities:

$$Z^{r+1}_{p,q} = \frac{d^{-1}(F_{p-(r+1)} C_{p+q-1}) \cap F_p C_{p+q}}{F_{p-1} C_{p+q}} = \ker d^r_{p,q} + B^r_{p,q}$$
$$B^{r+1}_{p,q} = \frac{d(F_{p+(r+1)-1} C_{p+q+1}) \cap F_p C_{p+q}}{F_{p-1} C_{p+q}} = \operatorname{Im} d^r_{p+r,q-r+1} + B^r_{p,q}$$

And $H_n(C) = \bigoplus_{p+q=n} \dfrac{\bigcap_r Z^r}{\sum_r B^r}$

1 Spectral Sequence and Convergence

1.1 Spectral Sequences and length-infty-filtration

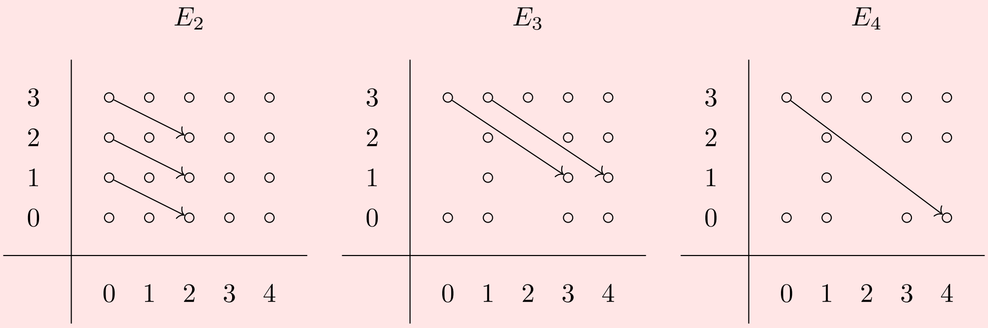
Definition 1.1. A **Spectral Sequence** (start at r_0-page ($r_0 \in \mathbb{N}$)) is a series of complexes (in some Abelian category) $\{E_r\}_{r=r_0}^{\infty}$ with **given differentials** $\{d_r : E_r \to E_r\}_{r=r_0}^{\infty}$ such that $E_{r+1} \cong H(E_r, d_r)$.

Note. Usually, a spectral sequence is bi-graded under following conventions:

- **Cohomological convention**

$$E_r = \bigoplus_{n\in\mathbb{Z}} \Big(\bigoplus_{p+q=n} E_r^{p,q} \Big)$$

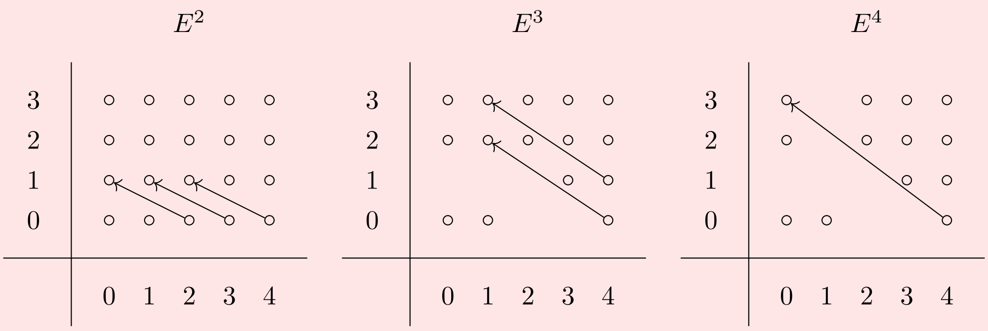
and d_r with bi-degree $(r, -r+1)$:



- **Homological convention**

$$E^r = \bigoplus_{n\in\mathbb{Z}} \Big(\bigoplus_{p+q=n} E_{p,q}^r \Big)$$

and d^r with bi-degree $(-r, r-1)$:



As we see, filtration of finite length on C could induce a spectral sequence $\{E^r, d^r\}_{r\in\mathbb{N}}$ which computes $H(C)$ for sufficient large r : $\mathrm{Tot}^{\oplus}(E^r) \cong H(C)$. But in practice, most of our filtrations have infinite length. When lengths are infinite, we need more constriants to insure that we could compute correct homologies from spectral sequences.

Definition 1.2. A **descending filtration** of X is a sequence of monomorphisms: $\{\cdots \rightarrowtail F^n X \rightarrowtail F^{n-1}X \rightarrowtail \cdots \rightarrowtail X\}_{n\in\mathbb{Z}}$.

Definition 1.3. A decreasing filtration $\{F^n X\}_{n\in\mathbb{Z}}$ of X (in some Abelian category) is called

- **exhaustive** if $\varinjlim_n X_n \cong X$.
- **Hausdorff** if $\varprojlim_n X_n \cong 0$.
- **complete** if $\varprojlim^1_n X_n \cong 0$.

Note. If $\{F^n X\}_{n\in\mathbb{Z}}$ is in $\mathbb{Z}-\mathrm{Mod}$, then we can topologize X by $\tau_X := \{x + F^n X \mid x \in X, n \in \mathbb{Z}\} \cup \{\emptyset, X\}$.

Then (X, τ_X) is Hausdorff topological space iff $\varprojlim_n X_n \cong 0$.

Recall that a Cauchy sequence in a topological group X is a sequence $x_- : \mathbb{N} \to X$ satisfy:

For any open neighborhood U of 0, there exists $N_U \in \mathbb{N}$ such that $\forall r, s \geq N_U \;.\; x_r^{-1} x_s \in U$.

$x \in X$ is a limit of a Cauchy sequence x_- if:

For any open neighborhood U of x, there exists $N_U \in \mathbb{N}$ such that $\forall r \geq N_U \;.\; x_r - x \in U$.

Take a Cauchy Sequence x_-, image of x_n in $\dfrac{X}{F^s X}$ is stable ($\operatorname{Im} x_r = \operatorname{Im} x_n, r \geq n$) for large enough n. By taking the stable image of x_- in $\dfrac{X}{F^s X}$, we defines a element $y^s \in \dfrac{X}{F^s X}$ and it passes to limit:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^s X & \longrightarrow & X & \longrightarrow & \dfrac{X}{F^s X} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F^{s+1} X & \longrightarrow & X & \longrightarrow & \dfrac{X}{F^{s+1} X} & \longrightarrow & 0 \end{array}$$

Apply $\varprojlim$, we obtain a exact sequence:

$$0 \longrightarrow \varprojlim_s F^s X \longrightarrow X \longrightarrow \varprojlim_s \frac{X}{F^s X} \longrightarrow \varprojlim_s^1 F^s X \longrightarrow 0$$

A limit of x_- exists iff these y^s could be lifted to X.

X is complete (all Cauchy sequence have limit) iff $\varprojlim_s^1 F^s X = 0$.

Definition 1.4. Let $\{E_{p,q}^r, d^r\}_{r=r_0}^{\infty}$ be a spectral sequence.

If forall $p, q \in \mathbb{Z}$ there exists $r(p,q) \in \mathbb{N}$ such that $r \geq r(p,q) \implies E_{p,q}^r \cong E_{p,q}^{r(p,q)}$, then we say $E_{p,q}^{\infty} := E_{p,q}^{r(p,q)}$ is the **limit term** of $\{E_{p,q}^r, d^r\}_{r=r_0}^{\infty}$, and $\{E_{p,q}^r, d^r\}_{r=r_0}^{\infty}$ **abuts to** E^{∞}.

If there exists N such that $r \geq N \implies \forall p, q \;.\; d_{p,q}^r = 0$, then we say E^r **collapses** at N-th page.