Fully Faithful descent

Cloudifold

The house of rising cat

29 January 2023

Glue

Theorem: glue sheaves

Let X be a topological space, $\{U_i\}_{i\in I}$ be a covering of X. If we have data:

- Sheaf F_i on U_i for each $i \in I$.
- Isomorphism $\phi_{i,j} : F_i|_{U_i\cap U_j} \to F_j|_{U_i\cap U_j}$ for each $i, j \in I$.

And them satisfy cocycle condition:

- $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,j,k}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf F on X such that $q_i : F|_{U_i} \cong F_i$ and $\phi_{i,j} = q_j \circ q_i^{-1}|_{U_i\cap U_j}$.

Skecth Proof:

Use product of germs which compatible with sheaf conditions.

Descent data on a family

Let S be a scheme, $U = \{X_i \to S\}_{i \in I}$ is a family of scheme morphisms.

And let $pr_i = X_0 \times_{Y_0} X_1 \times_{Y_1} \cdots \times_{Y_n} X_n \to X_i$

Similarly we have $pr_{ij}, pr_{ijk}, \dots$

Definition: descent data for quasi-coherent sheaves

A **descent datum** $(F_i, \phi_{i,j}$ for quasi-coherent sheaves with resp to family U is

- Quasi-coherent sheaf F_i on X_i for each $i \in I$.
- Isomorphism of quasi-coherent $O_{X_i \times_S X_j}$-modules $\phi_{i,j} : pr_0^* F_i \to pr_0^* F_j$ for each $i, j \in I$.

satisfying **cocycle condition**:

- $pr_{12}^* \phi_{j,k} \circ pr_{01}^* \phi_{i,j} = pr_{02}^* \phi_{i,j}$

$$\begin{tikzcd}
pr_0^* F_i \arrow[rr, "pr_{02}^* \phi_{i,k}"] \arrow[dr, "pr_{01}^* \phi_{i,j}"'] & & pr_2^* F_k \\
& pr_1^* F_j \arrow[ur, "pr_{12}^* \phi_{j,k}"'] &
\end{tikzcd}$$

Descent data on a family

A morphism $a : (F_i, \phi_{i,j}) \to (F_i', \phi_{i,j}')$ between descent data is morphisms $a_i : F_i \to G_i$ compatible with $\phi_{i,j}$:

$$\begin{tikzcd} pr_0^* F_i \arrow[r, "a_i"'] \arrow[d, "\phi"] & pr_0^* F_i' \arrow[d, "\phi"] \\ pr_1^* F_j \arrow[r, "a_j"'] & pr_1^* F_j' \end{tikzcd}$$

commutes.

Effective descent

Descent datum $(F_i, \phi_{i,j})$ for quasi-coherent sheaves with respect to the given covering is said to be **effective** if there exists a quasi-coherent sheaf F on S such that $(F_i, \phi_{i,j})$ is isomorphic to $(F|_{X_i}, canonical)$.

Family $U = \{X_i \to S\}_{i \in I}$ is said to be **have effective descent** if the category of descent datum (of quasi-coherent sheaves) on U $(Desc_U QCoh)$ is equivalent to $QCoh(S)$.

Fpqc descent, affine case

Main theorem:

If $R \to R'$ is faithfully flat ring map, then $\{Spec(R') \to Spec(R)\}$ have effective descent.

Sketch Proof:

Descent data M'_{const}, ϕ on resp $\{Spec(R') \to Spec(R)\} \simeq$ Cosimplicial diagrams like:

$$\begin{tikzcd} M' \arrow[r, shift left, "pr_0"] \arrow[r, shift right, "\phi^{-1} \circ pr_1"'] & M' \otimes_R R' \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & M' \otimes_R R' \otimes_R R' \end{tikzcd}$$

- Descend M' to R-Module: take kernel of diagram above.
- Ascend M to descent data: replace M' with R', ϕ with $id_{R'}$ in the diagram above, and apply $M \otimes_R -$.

Fpqc topology

Definition of fpqc covering

Let S be a scheme, a fpqc covering of S is a family of maps $\{f_i : X_i - > S\}_{i \in I}$ such that:

- each f_i flat and $\cup_{i \in I} f_i(X_i) = S$. (fp)
- for each affine open U in S, there exists finite set K and map $i : K \to I$ and affine opens $U_k \subset T_{i(k)}$, such that $U = \cup_K f_{i(k)}(U_k)$. (qc)

Example of fpqc covering

- Zariski open covering.
- $A \to B$ is faithfully flat iff $\{Spec(B) \to Spec(A)\}$ is a fpqc covering.
- Let $i_x : D(x) \to \mathbb{A}^2_k$ and $i_y : D(y) \to \mathbb{A}^2_k$, $\{i_x, i_y, Spec(k[[x, y]]) \to \mathbb{A}^2_k\}$ is a fpqc covering.

Fpqc topology

Proposition: fpqc topology is a site

- If $S' \to S$ is isomorphism, then $\{S' \to S\}$ is fpqc covering of S.
- If $S_i \to S_{i \in I}$ is fpqc covering, $\{S_{i,j} \to S_i\}_{j \in J_i}$ is fpqc covering for each $i \in I$, then $\{S_{i,j} \to S\}_{i \in I, j \in J_i}$ is fpqc covering.
- If $S_i \to S_{i \in I}$ is fpqc covering, $S' \to S$ is scheme map, then $S_i \times_S S' \to S'_{i \in I}$ is fpqc covering.

Proof:

Notice that composition of flat map is flat, choose affine opens and then choose quasi-compact opens.

Fpqc descent for QCoh

Theorem:

Every Fpqc covering have effective descent

Proof:

Choose affine covering multiple times. See stack 023T.

Descent of quasi-affine schemes

A scheme is **quasi-affine** if it is an open subscheme of an affine scheme and is quasi-compact.

A map $f : X \to S$ is **quasi-affine** if for every affine open U of S, $f^{-1}(S)$ is quasi-affine.

Theorem:

For a fpqc covering $U = \{X_i \to S\}$, the inclusion from category of quasi-affine S-scheme to category of descent data of quasi-affine schemes $(T_i, \phi_{i,j})$ is equivalence.

Proof:

Similar to fpqc descent for QCoh.

Stable under a descent datum

$X \to S$ is fpqc morphism, T is X-scheme, U is open subscheme of T, $\phi : p_0^* X \to p_1^* X$ is a descent datum. If ϕ induces descent datum on U, then U is called stable under ϕ.

Galois descent

Now we suppose k'/k is a finite Galois extension of fields, $G = Gal(k'/k)$. Left G-action on k' induces right G-action on $Spec(k')$.

Theorem:

Descent data on k'-scheme X' is equivalent to Right G-action on X' compatible with action on $Spec(k')$.

Proof:

$k' \otimes_k k' \cong \prod_{\sigma \in G} k'$. Calculation yields the results.

Another equivalent description

Giving a descent datum on X' is equivalent to giving a collection of k'-isomorphisms $\{f_\sigma : \sigma X \to X\}_{\sigma \in G}$ for $\sigma \in G$ satisfying cocycle condition $f_{\sigma\tau} = f_\sigma \cdot \sigma(f_\tau)$ for all $\sigma, \tau \in G$.

Corollaries:

- An isomorphism between varieties with descent data $X, \{f_\sigma\}_{\sigma \in G}$ and $Y, X, \{g_\sigma\}_{\sigma \in G}$ is k'-isomorphism $a : X \to Y$ satisfying $f_\sigma = a^{-1} g_\sigma \circ \sigma(a)$ forall $\sigma in G$.
- $U' \subset X'$ is stable under $\{f_\sigma\}_{\sigma \in G}$ iff $U' = f_\sigma(\sigma U')$ forall $\sigma \in G$.

Galois descent of quasi projective scheme

Theorem:

k'/k finite galois extension, X' is k' scheme with descent data $\{f_\sigma\}_{\sigma\in G}$. Then $X' = X_{k'}$ for a k-scheme X.

Sketch Proof:

It is equivalent to show X' can be covered by G-invarant quasi-affine opens. Fix embedding $X' \to \mathbb{P}^n_{k'}$, for every point x', choose a hypersurface H avoiding G-orbit of x'. Let $U' = X' - H$, $\cap_{\sigma\in G}\sigma(U')$ is quasi-affine open set contain x'.

Twists

Let X be quasi-affine k-variety, k'/k finite Galois extension, $G = Gal(k'/k)$.

Definition

A k'/k-**twist** of X is k-variety Y such that exists isomorphism $X_{k'} \cong Y_{k'}$.
A twist of X is k_{sp}/k-twist of X.

Theorem:

(Assume k'/k is finite) There is a bijection from k'/k-twists of X up to k-isomorphism to $H^1(G, Aut(X_{k'}))$.

Proof:

- k'/k-twists $\cong$ descent data of $X_{k'}$ $\cong$ 1-cocycles $G \to AutX_{k'}$
- k-isomorphic iff descent data isomorphic. Descent data isomorphic iff cocycle cohomologous.

Example

Rational points on a quadratic twist of an elliptic curve

k be a field, let E be a smooth elliptic curve over k defined by $y^2 = x^3 + ax + b$. Suppose $d \in k^\times/k^{\times 2}$, let $L = k(\sqrt{d})$, let σ be the nontrivial element in $Gal(L/k)$.
Let E' be elliptic curve defined by $dy^2 = x^3 + ax + b$, $E'(k) \cong \{P \in E(L), \sigma P = -P\}$.

Proof:

$E(L)$ is isomorphic to $E'(L)$ by $f : (x, y) \mapsto (x, \sqrt{d}y)$. $E'(k) = \{P \in E(L), \sigma f(P) = f(P)\}$, and notice that $\sigma f(P) = -f(\sigma P)$.

Torsors

Generalization of principal bundles.

Defintion

A **trivial torsor** in category C is:

- A object X.
- A G action on X
- A G-equivariant map $f : X \to Y$ with G act on Y trivially.

such that:

- Exists G-equivariant isomorphism $\phi : G \times Y \cong X$ such that $\phi \circ f = pr_1 : G \times Y \to Y$.

Defition

A X-torsor under G is an object X with G-action and an equivariant map $f : X \to Y$ with G trivial acts on Y such that:

- There exists a covering $\{Y_i \to Y\}$ such that for each i, $Y_i \times_Y X \to Y_i$ is trivial torsor.